

Error estimates in periodic homogenization with a non-homogeneous Dirichlet condition.

G. Griso

Laboratoire J.-L. Lions–CNRS, Boîte courrier 187, Université Pierre et Marie Curie,
4 place Jussieu, 75005 Paris, France, Email: griso@ann.jussieu.fr

Abstract

In this paper we investigate the homogenization problem with a non-homogeneous Dirichlet condition. Our aim is to give error estimates with boundary data in $H^{1/2}(\partial\Omega)$. The tools used are those of the unfolding method in periodic homogenization.

1 Introduction

We consider the following homogenization problem:

$$\phi^\varepsilon \in H^1(\Omega), \quad -\operatorname{div}(A_\varepsilon \nabla \phi^\varepsilon) = f \quad \text{in } \Omega, \quad \phi^\varepsilon = g \quad \text{on } \partial\Omega$$

where A_ε is a periodic matrix satisfying the usual condition of uniform ellipticity and where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)^1$. We know (see e.g. [9]) that the function ϕ^ε weakly converges in $H^1(\Omega)$ towards the solution Φ of the homogenized problem

$$\Phi \in H^1(\Omega), \quad -\operatorname{div}(\mathcal{A} \nabla \Phi) = f \quad \text{in } \Omega, \quad \Phi = g \quad \text{on } \partial\Omega$$

where \mathcal{A} is the homogenized matrix (see (4.4) and (4.5)). Using the results of [9] we can give an approximation of ϕ^ε belonging to $H^1(\Omega)$ and we easily obtain

$$\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \longrightarrow 0 \quad \text{strongly in } H^1(\Omega)$$

where \mathcal{Q}_ε is the *scale-splitting operator* (see also Subsection 2.4) and where the χ_i are the correctors (see (4.2)).

The aim of this paper is to give error estimates. Obviously, if we have $g \in H^{3/2}(\partial\Omega)$ and the appropriate assumptions on the boundary of the domain then we can apply the

¹The homogenization problem with a L^p boundary data is investigated in [3].

results obtained in [4], [12], [13], [14], [15] and [20] to deduce error estimates. All of them require that the function Φ belongs at least to $H^2(\Omega)$. Here, the solution Φ of the homogenized problem is only in $H^1(\Omega) \cap H_{loc}^2(\Omega)$. In this paper we must work with this lack of regularity; this is the main difficulty.

Our method to obtain error estimates (see [13], [14] and [15]) is mainly based on projection theorems. This is why we prove here two new projection theorems. In the second one, for a function $\phi \in H_0^1(\Omega)$ satisfying $1/\rho \nabla \phi \in L^2(\Omega; \mathbb{R}^n)$ -where $\rho(x)$ is the distance between x and the boundary of Ω - we obtain an upper bound for the distance between $\mathcal{T}_\varepsilon(\nabla \phi)$ and the space $\nabla H^1(\Omega) \oplus \nabla_y L^2(\Omega; H_{per}^1(Y))$ for an appropriate norm (see Section 3). Then, due to the result recalled in Subsection 7.1 of the Appendix we introduce a lifting in $H^1(\Omega)$ for the function $g \in H^{1/2}(\partial\Omega)$ and we show estimates in $H^1(\Omega)$ and in $H_{loc}^1(\Omega)$ using $\|g\|_{H^{1/2}(\partial\Omega)}$ and $\|g\|_{H^{-1/2}(\partial\Omega)}$. Afterwards, all the tools of the unfolding method (see [9]) and the results obtained in the first sections allow to derive the main results of the paper (Theorems 5.1 and 5.3). In both theorems we give L^2 error estimates and H_{loc}^1 error estimates. It is worth noting that the error estimates are only of order $\varepsilon^{1/2}$ while in [15] the obtained error was of order ε . We end the paper by investigating the case where the boundary data are strongly oscillating. A forthcoming paper we will be devoted to the homogenization problem with other strongly oscillating boundary data.

In Section 2 we introduce a few general notations, we also give some recalls on lemmas, definitions and results about the unfolding method in periodic homogenization; this complements the paper which presents the unfolding method (see [9]). Section 3 is devoted to the proofs of two new projection theorems which supplement the ones given in [14] and [15]. In Section 4, we recall the main results on the classical homogenization problem. In Section 5 we derive the error estimate results (Theorems 5.1 and 5.3) with a non-homogeneous Dirichlet condition and in the last section we investigate a first case of a strongly oscillating boundary data (Theorem 6.1). In the Appendix we introduce an operator from $H^{-1/2}(\partial\Omega)$ into $L^2(\Omega)$.

As general references on the homogenization theory we refer to [1], [4] and [12]. The reader is referred to [9], [11] and [12] for an introduction of the unfolding method in periodic homogenization. The following papers [5], [6], [7], [10] give various applications of the unfolding method in periodic homogenization. As far as the error estimates are concerned, we refer to [2], [4], [13], [14], [15], [18], [20] and [21].

Keywords: periodic homogenization, error estimate, non-homogeneous Dirichlet condition, unfolding method.

Mathematics Subject Classification (2000): 35B27, 65M15, 74Q15.

2 Preliminaries

2.1 Notations

- We denote by Ω a bounded domain in \mathbb{R}^n with a Lipschitz boundary.² Let $\rho(x)$ be the distance between $x \in \mathbb{R}^n$ and the boundary of Ω , we set

$$\tilde{\Omega}_\gamma = \left\{ x \in \Omega \mid \rho(x) < \gamma \right\} \quad \tilde{\tilde{\Omega}}_\gamma = \left\{ x \in \mathbb{R}^n \mid \rho(x) < \gamma \right\} \quad \gamma \in \mathbb{R}^{*+}.$$

- There exist constants a, A and γ_0 strictly positive and $M \geq 1$, a finite number N of local euclidian coordinate systems $(O_r; \mathbf{e}_{1r}, \dots, \mathbf{e}_{nr})$ and mappings $f_r : [-a, a]^{n-1} \rightarrow \mathbb{R}$, Lipschitz continuous with ratio M , $1 \leq r \leq N$, such that (see e.g. [16] or [17])

$$\begin{aligned} \partial\Omega &= \bigcup_{r=1}^N \left\{ x = x'_r + x_{nr}\mathbf{e}_{nr} \in \mathbb{R}^n \mid x'_r \in \Delta_a \text{ and } x_{nr} = f_r(x'_r) \right\}, \\ \text{where } x'_r &= x_{1r}\mathbf{e}_{1r} + \dots + x_{n-1r}\mathbf{e}_{n-1r}, \quad \Delta_a = \left\{ x'_r \mid x_{ir} \in]-a, a[, i \in \{1, \dots, n-1\} \right\} \\ \tilde{\Omega}_{\gamma_0} &\subset \bigcup_{r=1}^N \Omega_r \subset \Omega, \quad \Omega_r = \left\{ x \in \mathbb{R}^n \mid x'_r \in \Delta_a \text{ and } f_r(x'_r) < x_{nr} < f_r(x'_r) + A \right\} \\ \tilde{\tilde{\Omega}}_{\gamma_0} &\subset \bigcup_{r=1}^N \left\{ x \in \mathbb{R}^n \mid x'_r \in \Delta_a \text{ and } f_r(x'_r) - A < x_{nr} < f_r(x'_r) + A \right\} \\ \forall r \in \{1, \dots, N\}, \quad \forall x \in \Omega_r \quad \text{we have} \quad &\frac{1}{2M}(x_{nr} - f_r(x'_r)) \leq \rho(x) \leq x_{nr} - f_r(x'_r). \end{aligned} \tag{2.1}$$

- We set

$$\begin{aligned} Y &=]0, 1[^n, \quad \Xi_\varepsilon = \left\{ \xi \in \mathbb{Z}^n \mid \varepsilon(\xi + Y) \subset \Omega \right\}, \\ \hat{\Omega}_\varepsilon &= \text{interior} \left(\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \bar{Y}) \right), \quad \Lambda_\varepsilon = \Omega \setminus \hat{\Omega}_\varepsilon, \end{aligned}$$

where ε is a strictly positive real.

- We define

$$\begin{aligned} \star H_\rho^1(\Omega) &= \left\{ \phi \in L^2(\Omega) \mid \rho \nabla \phi \in L^2(\Omega; \mathbb{R}^n) \right\}, \\ \star L_{1/\rho}^2(\Omega) &= \left\{ \phi \in L^2(\Omega) \mid \phi/\rho \in L^2(\Omega) \right\}, \\ \star H_{1/\rho}^1(\Omega) &= \left\{ \phi \in H_0^1(\Omega) \mid \nabla \phi/\rho \in L^2(\Omega; \mathbb{R}^n) \right\}. \end{aligned}$$

We endow $H_\rho^1(\Omega)$ (resp. $H_{1/\rho}^1(\Omega)$) with the norm

$$\begin{aligned} \forall \phi \in H_\rho^1(\Omega), \quad \|\phi\|_\rho &= \|\phi\|_{L^2(\Omega)} + \|\rho \nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)} \\ (\text{resp. } \forall \phi \in H_{1/\rho}^1(\Omega), \quad \|\phi\|_{1/\rho} &= \|\nabla \phi/\rho\|_{L^2(\Omega; \mathbb{R}^n)}). \end{aligned}$$

²In Section 7.1 and those which follow, we will assume that Ω is a bounded domain of class $\mathcal{C}^{1,1}$ or an open bounded convex polygon ($n = 2$) or polyhedral ($n = 3$).

Remark 2.1. If ϕ belongs to $H_\rho^1(\Omega)$ then the function $\psi = \rho\phi$ is in $H_0^1(\Omega)$ and vice versa if the function ψ belongs to $H_0^1(\Omega)$ then $\phi = \psi/\rho$ is in $H_\rho^1(\Omega)$ since we have (see [8] or [19])

$$\forall \psi \in H_0^1(\Omega), \quad \|\psi/\rho\|_{L^2(\Omega)} \leq C \|\nabla \psi\|_{L^2(\Omega; \mathbb{R}^n)}. \quad (2.2)$$

The space \mathbb{R}^k ($k \geq 1$) is endowed with the standard basis $(\mathbf{e}_1, \dots, \mathbf{e}_k)$; the euclidian norm is denoted $|\cdot|$.

2.2 A characterization of the functions belonging to $H_{1/\rho}^1(\Omega)$

Observe first that if a function ϕ satisfies $\phi/\rho \in H_0^1(\Omega)$ then ϕ belongs to $H_{1/\rho}^1(\Omega)$. The reverse is true.

Lemma 2.2. Let Ω be a bounded open set with a Lipschitz boundary, we have

$$\phi \in H_{1/\rho}^1(\Omega) \iff \phi/\rho \in H_0^1(\Omega).$$

Furthermore there exists a constant which depends only on $\partial\Omega$ such that

$$\forall \phi \in H_{1/\rho}^1(\Omega) \quad \|\phi/\rho^2\|_{L^2(\Omega)} + \|\phi/\rho\|_{H^1(\Omega)} \leq C \|\phi\|_{1/\rho}. \quad (2.3)$$

Proof. Step 1. Let ϕ be in $H^1(]-a, a[^{n-1} \times]0, A[)$ ($a, A > 0$) satisfying $\frac{1}{x_n} \nabla \phi(x) \in L^2(]-a, a[^{n-1} \times]0, A[)$ and $\phi(x) = 0$ for a.e. x in $]-a, a[^{n-1} \times \{0\} \cup]-a, a[^{n-1} \times \{A\}$. We have

$$\int_{]-a, a[^{n-1} \times]0, A[} \frac{|\phi(x)|^2}{x_n^4} dx \leq \frac{1}{2} \int_{]-a, a[^{n-1} \times]0, A[} \frac{|\nabla \phi(x)|^2}{x_n^2} dx. \quad (2.4)$$

To prove (2.4), we choose $\eta > 0$ and we integrate by parts $\int_{]-a, a[^{n-1} \times]0, A[} \frac{|\phi(x)|^2}{(\eta + x_n)^4} dx$, then thanks to the identity relation $2bc \leq b^2 + c^2$ we obtain

$$\begin{aligned} \int_{]-a, a[^{n-1} \times]0, A[} \frac{|\phi(x)|^2}{(\eta + x_n)^4} dx &\leq \frac{1}{2} \int_{]-a, a[^{n-1} \times]0, A[} \frac{1}{(\eta + x_n)^2} \left| \frac{\partial \phi}{\partial x_n}(x) \right|^2 dx \\ &\leq \frac{1}{2} \int_{]-a, a[^{n-1} \times]0, A[} \frac{|\nabla \phi(x)|^2}{x_n^2} dx. \end{aligned}$$

Passing to the limit ($\eta \rightarrow 0$) it leads to (2.4).

Step 2. Let h be in $W^{1,\infty}(\Omega)$ such that

$$\begin{aligned} h(x) &\in [0, 1] \quad \text{if } x \in \Omega, \\ h(x) &= 1 \quad \text{if } \rho(x) \geq \gamma_0, \\ h(x) &= 0 \quad \text{if } \rho(x) \leq \gamma_0/2. \end{aligned}$$

Let ϕ be in $H_{1/\rho}^1(\Omega)$. The function $\phi h/\rho^4$ belongs to $H_0^1(\Omega)$, therefore as a consequence of the Poincaré's inequality we obtain

$$\begin{aligned} \int_{\Omega} \frac{|\phi(x)h(x)|^2}{\rho(x)^4} dx &\leq C \int_{\Omega} \left| \nabla \left(\frac{\phi(x)h(x)}{\rho(x)^4} \right) \right|^2 dx \leq C \int_{\Omega} (|\nabla \phi(x)|^2 + |\phi(x)|^2) dx \\ &\leq C \int_{\Omega} |\nabla \phi(x)|^2 dx \leq C \int_{\Omega} \frac{|\nabla \phi(x)|^2}{\rho(x)^2} dx. \end{aligned} \quad (2.5)$$

Then, due to the covering (2.1) of $\tilde{\Omega}_{\gamma_0}$, the inequality (2.4) and thanks to a simple change of variables we get

$$\int_{\Omega_r} \frac{|\phi(x)(1-h(x))|^2}{\rho(x)^4} dx \leq C \int_{\Omega_r} \frac{|\nabla(\phi(x)(1-h(x)))|^2}{\rho(x)^2} dx \leq C \int_{\Omega_r} \frac{|\nabla \phi(x)|^2 + |\phi(x)|^2}{\rho(x)^2} dx.$$

Since $\phi \in H_0^1(\Omega)$ the function ϕ/ρ belongs to $L^2(\Omega)$ and we have (2.2). Hence, adding these inequalities ($r = 1, \dots, N$) we obtain

$$\int_{\Omega} \frac{|\phi(x)(1-h(x))|^2}{\rho(x)^4} dx \leq C \int_{\Omega} \frac{|\nabla \phi(x)|^2}{\rho(x)^2} dx. \quad (2.6)$$

Finally $\phi/\rho^2 \in L^2(\Omega)$ and (2.5)-(2.6) lead to $\|\phi/\rho^2\|_{L^2(\Omega)} \leq C\|\phi\|_{1/\rho}$ and then (2.3). \square

2.3 Three lemmas

In the below lemma we give sharp estimates of a function on the boundary and in a neighborhood of the boundary.

Lemma 2.3. *Let Ω be a bounded open set with a Lipschitz boundary, there exists $\gamma_0 > 0$ (see Subsection 2.2) such that for any $\gamma \in]0, \gamma_0]$ and for any $\phi \in H^1(\Omega)$ we have*

$$\begin{aligned} \|\phi\|_{L^2(\partial\Omega)} &\leq \frac{C}{\gamma^{1/2}} (\|\phi\|_{L^2(\tilde{\Omega}_{\gamma})} + \gamma \|\nabla \phi\|_{L^2(\tilde{\Omega}_{\gamma}; \mathbb{R}^n)}), \\ \|\phi\|_{L^2(\tilde{\Omega}_{\gamma})} &\leq C(\gamma^{1/2} \|\phi\|_{L^2(\partial\Omega)} + \gamma \|\nabla \phi\|_{L^2(\tilde{\Omega}_{\gamma}; \mathbb{R}^n)}). \end{aligned} \quad (2.7)$$

The constants do not depend on γ .

Proof. Let ψ be in $H^1([-a, a]^{n-1} \times]0, A])$. For any $\eta \in]0, A[$ we have

$$\begin{aligned} \|\psi\|_{L^2([-a, a]^{n-1} \times \{0\})}^2 &\leq \frac{C}{\eta} \|\psi\|_{L^2([-a, a]^{n-1} \times]0, \eta])}^2 + C\eta \|\nabla \psi\|_{L^2([-a, a]^{n-1} \times]0, \eta]; \mathbb{R}^n)}^2, \\ \|\psi\|_{L^2([-a, a]^{n-1} \times]0, \eta])}^2 &\leq C\eta \|\psi\|_{L^2([-a, a]^{n-1} \times \{0\})}^2 + C\eta^2 \|\nabla \psi\|_{L^2([-a, a]^{n-1} \times]0, \eta]; \mathbb{R}^n)}^2. \end{aligned}$$

The constants do not depend on η . Now, let ϕ be in $H^1(\Omega)$. We use the above estimates, the covering of $\tilde{\Omega}_{\gamma_0}$ given by (2.1) and a simple change of variables to obtain (2.7). \square

Below we recall a classical extension lemma which is proved for example in [14] or which can be proved using the covering (2.1).

Lemma 2.4. *Let Ω be a bounded open set with a Lipschitz boundary, there exist $c_0 \geq 1$ (which depends only on the boundary of Ω) and a linear and continuous extension operator \mathcal{P} from $L^2(\Omega)$ into $L^2(\mathbb{R}^n)$ which also maps $H^1(\Omega)$ into $H^1(\mathbb{R}^n)$ such that*

$$\begin{aligned} \forall \phi \in L^2(\Omega), \quad \mathcal{P}(\phi)|_{\Omega} = \phi, \quad & \|\mathcal{P}(\phi)\|_{L^2(\mathbb{R}^n)} \leq C\|\phi\|_{L^2(\Omega)}, \\ & \|\mathcal{P}(\phi)\|_{L^2(\tilde{\Omega}_{c_0\gamma})} \leq C\|\phi\|_{L^2(\tilde{\Omega}_{c_0\gamma})} \end{aligned} \quad (2.8)$$

and moreover we have

$$\forall \phi \in H^1(\Omega), \quad \|\nabla \mathcal{P}(\phi)\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)} \leq C\|\nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)}.$$

From now on, if need be, a function ϕ belonging to $L^2(\Omega)$ (resp. $H^1(\Omega)$) will be extended to a function belonging to $L^2(\mathbb{R}^n)$ (resp. $H^1(\mathbb{R}^n)$) using the above lemma. The extension will be still denoted ϕ .

In the third lemma we show that a function in $H_0^1(\Omega)$ can be approached by functions vanishing outside of $\tilde{\Omega}_{6\sqrt{n}\varepsilon}$.

Lemma 2.5. *Let ϕ be in $H_0^1(\Omega)$, there exists $\phi_\varepsilon \in H^1(\mathbb{R}^n)$ satisfying*

$$\begin{aligned} \phi_\varepsilon(x) = 0 \quad & \text{for a.e. } x \notin \tilde{\Omega}_{6\sqrt{n}\varepsilon}, \\ \|\phi - \phi_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon\|\nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)}, \quad & \|\phi_\varepsilon\|_{H^1(\Omega)} \leq C\|\phi\|_{H^1(\Omega)}. \end{aligned} \quad (2.9)$$

Moreover, if $\phi \in H_{1/\rho}^1(\Omega)$ then we have

$$\|(\phi - \phi_\varepsilon)/\rho\|_{L^2(\Omega)} \leq C\varepsilon\|\nabla \phi\|_{1/\rho}, \quad \|\phi_\varepsilon\|_{1/\rho} \leq C\|\phi\|_{1/\rho}. \quad (2.10)$$

The constant C is independent of ε .

Proof. Let ϕ be in $H_0^1(\Omega)$. We define ϕ_ε by

$$\phi_\varepsilon(x) = \begin{cases} \frac{(\rho(x) - 6\sqrt{n}\varepsilon)^+}{\rho(x)} \phi(x) & \text{for a. e. } x \in \Omega, \\ 0 & \text{for a. e. } x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

where $\delta^+ = \max\{0, \delta\}$. The above function ϕ_ε belongs to $H^1(\mathbb{R}^n)$ and satisfies $\phi_\varepsilon = 0$ outside of $\tilde{\Omega}_{6\sqrt{n}\varepsilon}$. Then, due to the fact that ϕ/ρ belongs to $L^2(\Omega)$ and verifies $\|\phi/\rho\|_{L^2(\Omega)} \leq C\|\nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)}$ we obtain the estimates in (2.9). If $\phi \in H_{1/\rho}^1(\Omega)$ we use the estimate (2.3) to obtain (2.10). \square

2.4 Recalls and complements on the unfolding operators

In the sequel, we will make use of some definitions and results from [9] concerning the periodic unfolding method. For almost every $x \in \mathbb{R}^n$, there exists a unique element in \mathbb{Z}^n denoted $[x]$ such that

$$x = [x] + \{x\}, \quad \{x\} \in Y.$$

- *The unfolding operator \mathcal{T}_ε .*

For any $\phi \in L^1(\Omega)$, the function $\mathcal{T}_\varepsilon(\phi) \in L^1(\Omega \times Y)$ is given by

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \quad (2.11)$$

Since $\Lambda_\varepsilon \subset \widetilde{\Omega}_{\sqrt{n}\varepsilon}$, using Proposition 2.5 in [9] we get

$$\left| \int_{\Omega} \phi(x) dx - \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\phi)(x, y) dx dy \right| \leq \int_{\Lambda_\varepsilon} |\phi(x)| dx \leq \|\phi\|_{L^1(\widetilde{\Omega}_{\sqrt{n}\varepsilon})} \quad (2.12)$$

For $\phi \in L^2(\Omega)$ we have

$$\|\mathcal{T}_\varepsilon(\phi)\|_{L^2(\Omega)} \leq \|\phi\|_{L^2(\Omega)}. \quad (2.13)$$

We also have (see Proposition 2.5 in [9]) for $\phi \in H^1(\Omega)$ (resp. $\psi \in H_0^1(\Omega)$)

$$\begin{aligned} \|\mathcal{T}_\varepsilon(\phi) - \phi\|_{L^2(\widehat{\Omega}_\varepsilon \times Y)} &\leq C\varepsilon \|\nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)} \\ (\text{resp. } \|\mathcal{T}_\varepsilon(\psi) - \psi\|_{L^2(\Omega \times Y)} &\leq C\varepsilon \|\nabla \psi\|_{L^2(\Omega; \mathbb{R}^n)}). \end{aligned} \quad (2.14)$$

- *The local average operator \mathcal{M}_ε*

For $\phi \in L^1(\mathbb{R}^n)$, the function $\mathcal{M}_\varepsilon(\phi) \in L^\infty(\mathbb{R}^n)$ is defined by

$$\mathcal{M}_\varepsilon(\phi)(x) = \int_Y \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) dy \quad \text{for a.e. } x \in \mathbb{R}^n. \quad (2.15)$$

The value of $\mathcal{M}_\varepsilon(\phi)$ in the cell $\varepsilon(\xi + Y)$ ($\xi \in \mathbb{Z}^n$) will be denoted $\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)$. In [9] we proved the following results:

For $\phi \in L^2(\Omega)$ we have

$$\|\mathcal{M}_\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\|\phi\|_{L^2(\Omega)}, \quad \|\mathcal{M}_\varepsilon(\phi) - \phi\|_{H^{-1}(\Omega)} \leq C\varepsilon\|\phi\|_{L^2(\Omega)} \quad (2.16)$$

and for $\psi \in H_0^1(\Omega)$ (resp. $\phi \in H^1(\Omega)$) we have

$$\begin{aligned} \|\mathcal{M}_\varepsilon(\psi) - \psi\|_{L^2(\Omega)} &\leq C\varepsilon \|\nabla \psi\|_{L^2(\Omega; \mathbb{R}^n)} \\ (\text{resp. } \|\mathcal{M}_\varepsilon(\phi) - \phi\|_{L^2(\widehat{\Omega}_\varepsilon)} &\leq C\varepsilon \|\nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)}). \end{aligned} \quad (2.17)$$

Lemma 2.6. For $\phi \in H_\rho^1(\Omega)$ we have

$$\begin{aligned} \forall i \in \{1, \dots, n\}, \quad & \|\rho(\mathcal{M}_\varepsilon(\phi) - \phi)\|_{L^2(\Omega)} \leq C\varepsilon\|\phi\|_\rho, \\ & \|\rho(\phi(\cdot + \varepsilon\mathbf{e}_i) - \phi)\|_{L^2(\Omega)} \leq C\varepsilon\|\phi\|_\rho, \\ & \|\rho(\mathcal{M}_\varepsilon(\phi)(\cdot + \varepsilon\mathbf{e}_i) - \mathcal{M}_\varepsilon(\phi))\|_{L^2(\Omega)} \leq C\varepsilon\|\phi\|_\rho. \end{aligned} \quad (2.18)$$

For $\phi \in L_{1/\rho}^2(\Omega)$ we have

$$\|\mathcal{M}_\varepsilon(\phi) - \phi\|_{(H_\rho^1(\Omega))'} \leq C\varepsilon\|\phi/\rho\|_{L^2(\Omega)}. \quad (2.19)$$

The constants do not depend on ε .

Proof. Step 1. We prove (2.18)₁. Let ϕ be in $H_\rho^1(\Omega)$ and let $\varepsilon(\xi + Y)$ be a cell included in Ω .

Case 1: $\rho(\varepsilon\xi) \geq 2\sqrt{n}\varepsilon$. In this case, observing that

$$1 \leq \frac{\max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\}}{\min_{z \in \varepsilon(\xi+Y)} \{\rho(z)\}} \leq 3$$

and thanks to the Poincaré-Wirtinger's inequality we obtain

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx &\leq \left[\max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\} \right]^2 \int_{\varepsilon(\xi+Y)} |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx \\ &\leq \left[\max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\} \right]^2 C\varepsilon^2 \int_{\varepsilon(\xi+Y)} |\nabla \phi(x)|^2 dx \\ &\leq C\varepsilon^2 \int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\nabla \phi(x)|^2 dx. \end{aligned}$$

Case 2: $\rho(\varepsilon\xi) \leq 2\sqrt{n}\varepsilon$. In this case we have

$$\int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\varepsilon(\xi+Y)} |\phi(x)|^2 dx.$$

The cases 1 and 2 lead to

$$\int_{\widehat{\Omega}_\varepsilon} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(x) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\widehat{\Omega}_\varepsilon} ([\rho(x)]^2 |\nabla \phi(x)|^2 + |\phi(x)|^2) dx. \quad (2.20)$$

Then, since $\Lambda_\varepsilon \subset \widetilde{\Omega}_{\sqrt{n}\varepsilon}$ and thanks to (2.8) we get

$$\int_{\Lambda_\varepsilon} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(x) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\widetilde{\Omega}_{c_0\sqrt{n}\varepsilon}} |\phi(x)|^2 dx$$

which due to (2.20) gives (2.18)₁. Proceeding in the same way we obtain (2.18)₂ and (2.18)₃.

Step 2. We prove (2.19). Let ϕ be in $L^2_{1/\rho}(\Omega)$ and $\psi \in H^1_\rho(\Omega)$. We have

$$\int_{\widehat{\Omega}_\varepsilon} (\mathcal{M}_\varepsilon(\phi) - \phi)\psi = \int_{\widehat{\Omega}_\varepsilon} (\mathcal{M}_\varepsilon(\psi) - \psi)\phi.$$

Consequently we obtain

$$\begin{aligned} \left| \int_{\Omega} (\mathcal{M}_\varepsilon(\phi) - \phi)\psi - \int_{\Omega} (\mathcal{M}_\varepsilon(\psi) - \psi)\phi \right| &\leq \int_{\Lambda_\varepsilon} |(\mathcal{M}_\varepsilon(\phi) - \phi)\psi| + \int_{\Lambda_\varepsilon} |(\mathcal{M}_\varepsilon(\psi) - \psi)\phi| \\ &\leq C(\|\phi\|_{L^2(\Lambda_\varepsilon)} + \|\mathcal{M}_\varepsilon(\phi)\|_{L^2(\Lambda_\varepsilon)})\|\psi\|_{L^2(\Omega)}. \end{aligned}$$

The inclusion $\Lambda_\varepsilon \subset \widetilde{\Omega}_{\sqrt{n}\varepsilon}$, the fact that $\phi \in L^2_{1/\rho}(\Omega)$ and (2.8)₁-(2.18)₁ lead to

$$\int_{\Omega} (\mathcal{M}_\varepsilon(\phi) - \phi)\psi \leq C\varepsilon\|\phi/\rho\|_{L^2(\Omega)}\|\psi\|_\rho.$$

Hence we get (2.19). □

• *The scale-splitting operator \mathcal{Q}_ε .*

★ For $\phi \in L^1(\mathbb{R}^n)$, the function $\mathcal{Q}_\varepsilon(\phi) \in W^{1,\infty}(\mathbb{R}^n)$ is given by

$$\mathcal{Q}_\varepsilon(\phi)(x) = \sum_{\xi \in \mathbb{Z}^n} \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) H_{\varepsilon,\xi}(x) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where

$$\begin{aligned} H_{\varepsilon,\xi}(x) &= H\left(\frac{x - \varepsilon\xi}{\varepsilon}\right) \quad \text{with} \\ H(z) &= \begin{cases} (1 - |z_1|)(1 - |z_2|) \dots (1 - |z_n|) & \text{if } z \in [-1, 1]^n, \\ 0 & \text{if } z \in \mathbb{R}^n \setminus [-1, 1]^n. \end{cases} \end{aligned}$$

Below we recall some results about \mathcal{Q}_ε proved in [9] and [15].

★ For $\phi \in L^2(\mathbb{R}^n)$ we have

$$\|\mathcal{Q}_\varepsilon(\phi)\|_{L^2(\mathbb{R}^n)} \leq C\|\phi\|_{L^2(\mathbb{R}^n)}, \quad \|\nabla \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \leq \frac{C}{\varepsilon}\|\phi\|_{L^2(\mathbb{R}^n)} \quad (2.21)$$

and

$$\mathcal{Q}_\varepsilon(\phi) \longrightarrow \phi \quad \text{strongly in } L^2(\mathbb{R}^n).$$

★ For $\phi \in H^1(\mathbb{R}^n)$ we have

$$\begin{aligned} \|\nabla \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} &\leq C\|\nabla \phi\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}, \\ \|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\mathbb{R}^n)} &\leq C\varepsilon\|\nabla \phi\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \end{aligned} \quad (2.22)$$

and

$$\mathcal{Q}_\varepsilon(\phi) \longrightarrow \phi \quad \text{strongly in } H^1(\mathbb{R}^n). \quad (2.23)$$

★ For $\phi \in L^2(\mathbb{R}^n)$ and $\chi \in L^2(Y)$ we have $\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \in L^2(\mathbb{R}^n)$, $\nabla \mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \in L^2(\mathbb{R}^n)$ and

$$\begin{aligned} \|\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\mathbb{R}^n)} &\leq C\|\phi\|_{L^2(\mathbb{R}^n)}\|\chi\|_{L^2(Y)}, \\ \|\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\tilde{\Omega}_{\sqrt{n\varepsilon}})} &\leq C\|\phi\|_{L^2(\tilde{\Omega}_{3\sqrt{n\varepsilon}})}\|\chi\|_{L^2(Y)}. \end{aligned} \quad (2.24)$$

Moreover, if $\phi \in H^1(\mathbb{R}^n)$ then we have

$$\begin{aligned} \|(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi))\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\mathbb{R}^n)} &\leq C\varepsilon\|\nabla\phi\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}\|\chi\|_{L^2(Y)}, \\ \|\nabla\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} &\leq C\|\nabla\phi\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}\|\chi\|_{L^2(Y)}, \\ \|\nabla\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\tilde{\Omega}_{\sqrt{n\varepsilon}};\mathbb{R}^n)} &\leq C\|\nabla\phi\|_{L^2(\tilde{\Omega}_{3\sqrt{n\varepsilon}};\mathbb{R}^n)}\|\chi\|_{L^2(Y)}. \end{aligned} \quad (2.25)$$

Lemma 2.7. For $\phi \in H_\rho^1(\Omega)$ we have

$$\|\rho(\mathcal{Q}_\varepsilon(\phi) - \phi)\|_{L^2(\Omega)} \leq C\varepsilon\|\phi\|_\rho \quad (2.26)$$

For $\phi \in H_{1/\rho}^1(\Omega)$ and ϕ_ε given by Lemma 2.5 we have

$$\begin{aligned} \|\mathcal{Q}_\varepsilon(\phi_\varepsilon)\|_{1/\rho} &\leq C\|\phi\|_{1/\rho}, \quad \|(\phi - \mathcal{Q}_\varepsilon(\phi_\varepsilon))/\rho\|_{L^2(\Omega)} \leq C\varepsilon\|\phi\|_{1/\rho}, \\ \forall \mathbf{i} = i_1\mathbf{e}_1 + \dots + i_n\mathbf{e}_n, \quad (i_1, \dots, i_n) &\in \{0, 1\}^n \\ \|(\mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot + \varepsilon\mathbf{i}) - \mathcal{M}_\varepsilon(\phi_\varepsilon))/\rho\|_{L^2(\Omega)} &\leq C\varepsilon\|\phi\|_{1/\rho}. \end{aligned} \quad (2.27)$$

For $\phi \in L^2(\mathbb{R}^n)$ and $\chi \in L^2(Y)$

$$\|(\mathcal{M}_\varepsilon(\rho\phi) - \rho\mathcal{M}_\varepsilon(\phi))\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon\|\phi\|_{L^2(\mathbb{R}^n)}\|\chi\|_{L^2(Y)}. \quad (2.28)$$

For $\phi \in H_\rho^1(\Omega)$ and $\chi \in L^2(Y)$

$$\begin{aligned} \|\rho(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi))\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\Omega)} &\leq C\varepsilon\|\phi\|_\rho\|\chi\|_{L^2(Y)}, \\ \|\rho\nabla\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\Omega)} &\leq C\|\phi\|_\rho\|\chi\|_{L^2(Y)}. \end{aligned} \quad (2.29)$$

The constants do not depend on ε .

Proof. Step 1. Let ϕ be in $H_\rho^1(\Omega)$. We first prove

$$\|\rho(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi))\|_{L^2(\Omega)} \leq C\varepsilon\|\phi\|_\rho. \quad (2.30)$$

To do that, we proceed as in the proof of (2.18)₁. Let $\varepsilon(\xi + Y)$ be a cell included in Ω .

Case 1: $\rho(\varepsilon\xi) \geq 3\sqrt{n\varepsilon}$. In this case we have

$$1 \leq \frac{\max_{z \in \varepsilon(\xi+Y)}\{\rho(z)\}}{\min_{z \in \varepsilon(\xi+2Y)}\{\rho(z)\}} \leq 4 \quad \text{and} \quad 1 \leq \frac{\max_{z \in \varepsilon(\xi+2Y)}\{\rho(z)\}}{\min_{z \in \varepsilon(\xi+Y)}\{\rho(z)\}} \leq \frac{5}{2}.$$

By definition of $\mathcal{Q}_\varepsilon(\phi)$ we deduce that

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2 dx &\leq \left[\max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\} \right]^2 \int_{\varepsilon(\xi+Y)} |\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2 dx \\ &\leq \left[\max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\} \right]^2 C\varepsilon^2 \int_{\varepsilon(\xi+2Y)} |\nabla \phi(x)|^2 dx \\ &\leq C\varepsilon^2 \int_{\varepsilon(\xi+2Y)} [\rho(x)]^2 |\nabla \phi(x)|^2 dx. \end{aligned}$$

Case 2: $\rho(\varepsilon\xi) \leq 3\sqrt{n}\varepsilon$. Then again by definition of $\mathcal{Q}_\varepsilon(\phi)$ we get

$$\int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2 dx \leq C\varepsilon^2 \int_{\varepsilon(\xi+2Y)} |\phi(x)|^2 dx.$$

As a consequence of the cases 1 and 2 we get

$$\int_{\widehat{\Omega}_\varepsilon} [\rho(x)]^2 |\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(x)|^2 dx \leq C\varepsilon^2 \int_{\Omega} ([\rho(x)]^2 |\nabla \phi(x)|^2 + |\phi(x)|^2) dx. \quad (2.31)$$

Furthermore we have

$$\int_{\Lambda_\varepsilon} [\rho(x)]^2 |\mathcal{Q}_\varepsilon(\phi)(x)|^2 dx \leq C\varepsilon^2 \int_{\Lambda_\varepsilon} |\mathcal{Q}_\varepsilon(\phi)(x)|^2 dx \leq C\varepsilon^2 \int_{\Omega} |\phi(x)|^2 dx$$

which with (2.31) lead to (2.30). Then as a consequence of (2.18)₁ and (2.30) we get (2.26).

Step 2. We prove (2.27)₁. Let ϕ be in $H_{1/\rho}^1(\Omega)$ and ϕ_ε given by Lemma 2.5. Due to the fact that $\phi_\varepsilon(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus \widetilde{\widetilde{\Omega}}_{6\sqrt{n}\varepsilon}$, hence $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) = 0$ for all $x \in \Omega$ such that $\rho(x) \leq 4\sqrt{n}\varepsilon$. Again we take a cell $\varepsilon(\xi+Y)$ included in Ω such that $\rho(\varepsilon\xi) \geq 3\sqrt{n}\varepsilon$. The values taken by $\mathcal{Q}_\varepsilon(\phi_\varepsilon)$ in the cell $\varepsilon(\xi+Y)$ depend only on the values of ϕ_ε in $\varepsilon(\xi+2Y)$. Then we have

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} \frac{1}{[\rho(x)]^2} |\nabla \mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)|^2 dx &\leq \frac{C}{[\min_{x \in \varepsilon(\xi+Y)} \{\rho(x)\}]^2} \int_{\varepsilon(\xi+2Y)} |\nabla \phi_\varepsilon(x)|^2 dx \\ &\leq C \frac{[\max_{x \in \varepsilon(\xi+2Y)} \{\rho(x)\}]^2}{[\min_{x \in \varepsilon(\xi+Y)} \{\rho(x)\}]^2} \int_{\varepsilon(\xi+2Y)} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx \leq C \int_{\varepsilon(\xi+2Y)} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx. \end{aligned}$$

Adding all these inequalities gives

$$\int_{\widetilde{\widetilde{\Omega}}_{4\sqrt{n}\varepsilon}} \frac{1}{[\rho(x)]^2} |\nabla \mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)|^2 dx \leq C \int_{\Omega} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx$$

Since $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) = 0$ for every $x \in \Omega$ such that $\rho(x) \leq 4\sqrt{n}\varepsilon$, we get $\|\mathcal{Q}_\varepsilon(\phi_\varepsilon)\|_{1/\rho} \leq C\|\phi_\varepsilon\|_{1/\rho}$. We conclude using (2.10)₂.

Step 3. Now we prove (2.27)₂. Again we consider a cell $\varepsilon(\xi + Y)$ included in Ω such that $\rho(\varepsilon\xi) \geq 3\sqrt{n}\varepsilon$. We have

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} \frac{1}{[\rho(x)]^2} |\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) - \phi_\varepsilon(x)|^2 dx &\leq \frac{C}{[\min_{x \in \varepsilon(\xi+Y)} \{\rho(x)\}]^2} \int_{\varepsilon(\xi+Y)} |\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) - \phi_\varepsilon(x)|^2 dx \\ &\leq \frac{C}{[\min_{x \in \varepsilon(\xi+Y)} \{\rho(x)\}]^2} \sum_{\mathbf{i} \in \{0,1\}^n} \int_{\varepsilon(\xi+\mathbf{i}+Y)} |\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi + \varepsilon\mathbf{i}) - \phi_\varepsilon(x)|^2 dx \\ &\leq C\varepsilon^2 \frac{[\max_{z \in \varepsilon(\xi+2Y)} \{\rho(z)\}]^2}{[\min_{z \in \varepsilon(\xi+Y)} \{\rho(z)\}]^2} \int_{\varepsilon(\xi+2Y)} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx \leq C\varepsilon^2 \int_{\varepsilon(\xi+2Y)} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx. \end{aligned}$$

Hence we get

$$\int_{\widehat{\Omega}_{4\sqrt{n}\varepsilon}} \frac{1}{[\rho(x)]^2} |\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) - \phi_\varepsilon(x)|^2 dx \leq C\varepsilon^2 \int_{\Omega} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx$$

The above estimate and the fact that $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) - \phi_\varepsilon(x) = 0$ for a.e. $x \in \Omega$ such that $\rho(x) \leq 4\sqrt{n}\varepsilon$ yield $\|(\phi_\varepsilon - \mathcal{Q}_\varepsilon(\phi_\varepsilon))/\rho\|_{L^2(\Omega)} \leq C\varepsilon\|\phi_\varepsilon\|_{1/\rho}$. We conclude using both estimates in (2.10).

Proceeding as in the Steps 2 and 3 we obtain (2.27)₃, (2.28) and (2.29). \square

3 Two new projection theorems

Theorem 3.1. *Let ϕ be in $H_{1/\rho}^1(\Omega)$. There exists $\widehat{\psi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega))$ such that*

$$\begin{cases} \|\widehat{\psi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C\{\|\phi\|_{L^2(\Omega)} + \varepsilon\|\nabla \phi\|_{[L^2(\Omega)]^n}\} \\ \|\mathcal{T}_\varepsilon(\phi) - \widehat{\psi}_\varepsilon\|_{H^1(Y; (H_\rho^1(\Omega))')} \leq C\varepsilon(\|\phi/\rho\|_{L^2(\Omega)} + \varepsilon\|\phi\|_{1/\rho}). \end{cases} \quad (3.1)$$

The constants depend only on n and $\partial\Omega$.

Proof. Here, we proceed as in the proof of the Proposition 3.3 of [5]. We first reintroduce the open sets $\widehat{\Omega}_{\varepsilon,i}$ and the unfolding operators $\mathcal{T}_{\varepsilon,i}$. We set

$$\widehat{\Omega}_{\varepsilon,i} = \widehat{\Omega}_\varepsilon \cap (\widehat{\Omega}_\varepsilon - \varepsilon\mathbf{e}_i), \quad K_i = \text{interior}(\overline{Y} \cup (\mathbf{e}_i + \overline{Y})), \quad i \in \{1, \dots, n\}.$$

The unfolding operator $\mathcal{T}_{\varepsilon,i}$ from $L^2(\Omega)$ into $L^2(\Omega \times K_i)$ is defined by

$$\forall \psi \in L^2(\Omega), \quad \mathcal{T}_{\varepsilon,i}(\psi)(x, y) = \begin{cases} \psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for } x \in \widehat{\Omega}_{\varepsilon,i} \text{ and for a.e. } y \in K_i, \\ 0 & \text{for } x \in \Omega \setminus \widehat{\Omega}_{\varepsilon,i} \text{ and for a.e. } y \in K_i. \end{cases}$$

The restriction of $\mathcal{T}_{\varepsilon,i}(\psi)$ to $\widehat{\Omega}_{\varepsilon,i} \times Y$ is equal to $\mathcal{T}_\varepsilon(\psi)$.

Step 1. Let us first take $\phi \in L^2_{1/\rho}(\Omega)$. We set $\psi = \frac{1}{\rho}\phi$ and we evaluate the difference $\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \cdot + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)$ in $L^2(Y; (H^1_\rho(\Omega))')$. For any $\Psi \in H^1_\rho(\Omega)$ a change of variables gives for a. e. $y \in Y$

$$\begin{aligned} \int_{\Omega} \mathcal{T}_{\varepsilon,i}(\phi)(x, y + \mathbf{e}_i) \Psi(x) dx &= \int_{\widehat{\Omega}_{\varepsilon,i}} \mathcal{T}_{\varepsilon}(\phi)(x + \varepsilon \mathbf{e}_i, y) \Psi(x) dx \\ &= \int_{\widehat{\Omega}_{\varepsilon,i} + \varepsilon \mathbf{e}_i} \mathcal{T}_{\varepsilon}(\phi)(x, y) \Psi(x - \varepsilon \mathbf{e}_i) dx. \end{aligned}$$

Then we obtain for a. e. $y \in Y$

$$\begin{aligned} & \left| \int_{\Omega} \{ \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y) \} \Psi - \int_{\widehat{\Omega}_{\varepsilon,i}} \mathcal{T}_{\varepsilon}(\psi)(\cdot, y) \rho \{ \Psi(\cdot - \varepsilon \mathbf{e}_i) - \Psi \} \right| \\ & \leq \left| \int_{\widehat{\Omega}_{\varepsilon,i}} \mathcal{T}_{\varepsilon}(\psi)(\cdot, y) (\mathcal{T}_{\varepsilon}(\rho) - \rho) \{ \Psi(\cdot - \varepsilon \mathbf{e}_i) - \Psi \} \right| + C \| \mathcal{T}_{\varepsilon}(\phi)(\cdot, y) \|_{L^2(\tilde{\Omega}_{2\sqrt{n\varepsilon}})} \| \Psi \|_{L^2(\tilde{\Omega}_{2\sqrt{n\varepsilon}})}. \end{aligned}$$

From (2.18)₂ we obtain

$$\| \rho(\Psi(\cdot - \varepsilon \mathbf{e}_i) - \Psi) \|_{L^2(\widehat{\Omega}_{\varepsilon,i})} \leq C\varepsilon \| \Psi \|_{\rho} \quad \forall i \in \{1, \dots, n\}.$$

We have

$$\| \mathcal{T}_{\varepsilon}(\rho) - \rho \|_{L^\infty(\Omega)} \leq C\varepsilon. \quad (3.2)$$

The above inequalities lead to

$$\begin{aligned} & < \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y), \Psi >_{(H^1_\rho(\Omega))', H^1_\rho(\Omega)} \\ & = \int_{\Omega} \{ \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y) \} \Psi \\ & \leq C\varepsilon \| \Psi \|_{\rho} \| \mathcal{T}_{\varepsilon}(\psi)(\cdot, y) \|_{L^2(\Omega)} + C\varepsilon \| \Psi \|_{L^2(\Omega)} \| \mathcal{T}_{\varepsilon}(\psi)(\cdot, y) \|_{L^2(\Omega)} \\ & + C \| \mathcal{T}_{\varepsilon}(\phi)(\cdot, y) \|_{L^2(\tilde{\Omega}_{2\sqrt{n\varepsilon}})} \| \Psi \|_{L^2(\tilde{\Omega}_{2\sqrt{n\varepsilon}})}. \end{aligned}$$

Therefore, for a.e. $y \in Y$ we have

$$\| \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y) \|_{(H^1_\rho(\Omega))'} \leq C\varepsilon \| \mathcal{T}_{\varepsilon}(\psi)(\cdot, y) \|_{L^2(\Omega)} + C \| \mathcal{T}_{\varepsilon}(\phi)(\cdot, y) \|_{L^2(\tilde{\Omega}_{2\sqrt{n\varepsilon}})}$$

which leads to the following estimate of the difference between $\mathcal{T}_{\varepsilon,i}(\phi)|_{\Omega \times Y}$ and one of its translated :

$$\begin{aligned} \| \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \cdot + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi) \|_{L^2(Y; (H^1_\rho(\Omega))')} &\leq C\varepsilon \| \phi/\rho \|_{L^2(\Omega)} + C \| \phi \|_{L^2(\tilde{\Omega}_{2\sqrt{n\varepsilon}})} \\ &\leq C\varepsilon \| \phi/\rho \|_{L^2(\Omega)}. \end{aligned} \quad (3.3)$$

The constant depends only on the boundary of Ω .

Step 2. Let $\phi \in H_{1/\rho}^1(\Omega)$. The above estimate (3.3) applied to ϕ and its partial derivatives give

$$\begin{aligned} \|\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \dots + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)\|_{L^2(Y; (H_\rho^1(\Omega))')} &\leq C\varepsilon \|\phi/\rho\|_{L^2(\Omega)} \\ \|\mathcal{T}_{\varepsilon,i}(\nabla\phi)(\cdot, \dots + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\nabla\phi)\|_{[L^2(Y; (H_\rho^1(\Omega))')^n]} &\leq C\varepsilon \|\phi\|_{1/\rho}. \end{aligned}$$

which in turn lead to (we recall that $\nabla_y(\mathcal{T}_{\varepsilon,i}(\phi)) = \varepsilon\mathcal{T}_{\varepsilon,i}(\nabla\phi)$).

$$\|\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \dots + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)\|_{H^1(Y; (H_\rho^1(\Omega))')} \leq C\varepsilon (\|\phi/\rho\|_{L^2(\Omega)} + \varepsilon\|\phi\|_{1/\rho}).$$

From these inequalities, for any $i \in \{1, \dots, n\}$, we deduce the estimate of the difference of the traces of $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(\cdot, y)$ on the faces Y_i and $\mathbf{e}_i + Y_i$

$$\|\mathcal{T}_\varepsilon(\phi)(\cdot, \dots + \mathbf{e}_i) - \mathcal{T}_\varepsilon(\phi)\|_{H^{1/2}(Y_i; (H_\rho^1(\Omega))')} \leq C\varepsilon (\|\phi/\rho\|_{L^2(\Omega)} + \varepsilon\|\phi\|_{1/\rho}). \quad (3.4)$$

It measures the periodic defect of $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(\cdot, y)$. We decompose $\mathcal{T}_\varepsilon(\phi)$ into the sum of an element belonging to $H_{per}^1(Y; L^2(\Omega))$ and an element belonging to $(H^1(Y; L^2(\Omega)))^\perp$ (the orthogonal of $H_{per}^1(Y; L^2(\Omega))$ in $H^1(Y; L^2(\Omega))$, see [5])

$$\mathcal{T}_\varepsilon(\phi) = \widehat{\psi}_\varepsilon + \overline{\phi}_\varepsilon, \quad \widehat{\psi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega)), \quad \overline{\phi}_\varepsilon \in (H^1(Y; L^2(\Omega)))^\perp. \quad (3.5)$$

The function $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(\cdot, y)$ takes its values in a finite dimensional space,

$$\overline{\phi}_\varepsilon(\cdot, \dots) = \sum_{\xi \in \Xi_\varepsilon} \overline{\phi}_{\varepsilon, \xi}(\cdot) \chi_\xi(\cdot)$$

where $\chi_\xi(\cdot)$ is the characteristic function of the cell $\varepsilon(\xi + Y)$ and where $\overline{\phi}_{\varepsilon, \xi}(\cdot) \in (H^1(Y))^\perp$ (the orthogonal of $H_{per}^1(Y)$ in $H^1(Y)$, see [5]). The decomposing (3.5) is the same in $H^1(Y; (H_\rho^1(\Omega))')$ and we have

$$\|\widehat{\psi}_\varepsilon\|_{H^1(Y; L^2(\Omega))}^2 + \|\overline{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))}^2 = \|\mathcal{T}_\varepsilon(\phi)\|_{H^1(Y; L^2(\Omega))}^2 \leq C \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla\phi\|_{[L^2(\Omega)]^n} \}^2$$

which is the first inequality in (3.1) and the estimate of $\overline{\phi}_\varepsilon$ in $H^1(Y; L^2(\Omega))$. From Theorem 2.2 of [5] and (3.4) we obtain a finer estimate of $\overline{\phi}_\varepsilon$ in $H^1(Y; (H_\rho^1(\Omega))')$

$$\|\overline{\phi}_\varepsilon\|_{H^1(Y; (H_\rho^1(\Omega))')} \leq C\varepsilon (\|\phi/\rho\|_{L^2(\Omega)} + \varepsilon\|\phi\|_{1/\rho}).$$

It is the second inequality in (3.1). □

Theorem 3.2. *For any $\phi \in H_{1/\rho}^1(\Omega)$, there exists $\widehat{\phi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega))$ such that*

$$\begin{aligned} \|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} &\leq C \|\nabla\phi\|_{[L^2(\Omega)]^n}, \\ \|\mathcal{T}_\varepsilon(\nabla\phi) - \nabla\phi - \nabla_y \widehat{\phi}_\varepsilon\|_{[L^2(Y; (H_\rho^1(\Omega))')^n]} &\leq C\varepsilon \|\phi\|_{1/\rho}. \end{aligned} \quad (3.6)$$

The constants depend only on $\partial\Omega$.

Proof. Let ϕ be in $H_{1/\rho}^1(\Omega)$ and $\psi = \phi/\rho \in H_0^1(\Omega)$. The function ϕ is extended by 0 outside of Ω . We decompose ϕ as

$$\phi = \Phi + \varepsilon \underline{\phi}, \quad \text{where } \Phi = \mathcal{Q}_\varepsilon(\phi_\varepsilon) \quad \text{and} \quad \underline{\phi} = \frac{1}{\varepsilon} \left(\phi - \mathcal{Q}_\varepsilon(\phi_\varepsilon) \right)$$

where ϕ_ε is given by Lemma 2.5. We have Φ and $\underline{\phi} \in H_0^1(\Omega)$ and due to (2.27) we get the following estimates:

$$\|\Phi\|_{1/\rho} + \varepsilon \|\underline{\phi}\|_{1/\rho} + \|\underline{\phi}/\rho\|_{L^2(\Omega)} \leq C \|\phi\|_{1/\rho}. \quad (3.7)$$

The projection Theorem 3.1 applied to $\underline{\phi} \in H_{1/\rho}^1(\Omega)$ gives an element $\widehat{\phi}_\varepsilon$ in $H_{per}^1(Y; L^2(\Omega))$ such that

$$\begin{aligned} \|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} &\leq C \|\phi\|_{1/\rho}, \\ \|\mathcal{T}_\varepsilon(\underline{\phi}) - \widehat{\phi}_\varepsilon\|_{H^1(Y; (H_\rho^1(\Omega))')} &\leq C\varepsilon \|\phi\|_{1/\rho}. \end{aligned} \quad (3.8)$$

Now we evaluate $\|\mathcal{T}_\varepsilon(\nabla \Phi) - \nabla \Phi\|_{[L^2(Y; (H_\rho^1(\Omega))')]^n}$.

From (2.19), (2.27)₁ and (3.7) we get

$$\|\nabla \Phi - \mathcal{M}_\varepsilon(\nabla \Phi)\|_{(H_\rho^1(\Omega; \mathbb{R}^n))'} \leq C\varepsilon \|\phi\|_{1/\rho}. \quad (3.9)$$

We set

$$\begin{aligned} H^{(1)}(z) &= \begin{cases} (1 - |z_2|)(1 - |z_3|) \dots (1 - |z_n|) & \text{if } z = (z_1, z_2, \dots, z_n) \in [-1, 1]^n, \\ 0 & \text{if } z \in \mathbb{R}^n \setminus [-1, 1]^n. \end{cases} \\ \mathbf{I} &= \left\{ \mathbf{i} \mid \mathbf{i} = i_2 \mathbf{e}_2 + \dots + i_n \mathbf{e}_n, \quad (i_2, \dots, i_n) \in \{0, 1\}^{n-1} \right\} \end{aligned}$$

For $\xi \in \mathbb{Z}^n$ and for every $(x, y) \in \varepsilon(\xi + Y) \times Y$ we have

$$\begin{aligned} \mathcal{T}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right)(x, y) &= \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{e}_1 + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))}{\varepsilon} H^{(1)}(y - \mathbf{i}) \\ \mathcal{M}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right)(\varepsilon \xi) &= \frac{1}{2^{n-1}} \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{e}_1 + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))}{\varepsilon}. \end{aligned}$$

Now, let us take $\psi \in H_\rho^1(\Omega)$. We recall that $\phi_\varepsilon(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus \widetilde{\widetilde{\Omega}}_{6\sqrt{n}\varepsilon}$, hence $\Phi(x) = 0$ for all $x \in \mathbb{R}^n \setminus \widetilde{\widetilde{\Omega}}_{3\sqrt{n}\varepsilon}$. As a first consequence $\mathcal{M}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right)$ in Λ_ε .

For $y \in Y$ we have

$$\begin{aligned} \langle \mathcal{T}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right), \psi \rangle_{(H_\rho^1(\Omega))', H_\rho^1(\Omega)} &= \int_\Omega \left\{ \mathcal{T}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right) \right\} \psi \\ &= \int_{\widehat{\Omega}_\varepsilon} \left\{ \mathcal{T}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial \Phi}{\partial x_1}\right) \right\} \mathcal{M}_\varepsilon(\psi). \end{aligned}$$

We have

$$\begin{aligned}
\int_{\widehat{\Omega}_\varepsilon} \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right) \mathcal{M}_\varepsilon(\psi) &= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\varepsilon\xi) \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi) \\
&= \frac{\varepsilon^n}{2^{n-1}} \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{e}_1 + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))}{\varepsilon} \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi) \\
&= \frac{\varepsilon^n}{2^{n-1}} \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\psi)(\varepsilon(\xi - \mathbf{e}_1)) - \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)}{\varepsilon} \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\widehat{\Omega}_\varepsilon} \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\cdot, y) \mathcal{M}_\varepsilon(\psi) \\
&= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \left[\frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{e}_1 + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))}{\varepsilon} \right] H^{(1)}(y - \mathbf{i}) \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi) \\
&= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\psi)(\varepsilon(\xi - \mathbf{e}_1)) - \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)}{\varepsilon} H^{(1)}(y - \mathbf{i}) \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))
\end{aligned}$$

Due to the fact that $\phi_\varepsilon(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus \widetilde{\Omega}_{6\sqrt{n}\varepsilon}$, in the above summations we only need to take $\xi \in \Xi_\varepsilon$ satisfying $\rho(\varepsilon\xi) \geq 3\sqrt{n}\varepsilon$. Hence

$$\begin{aligned}
&< \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right), \psi >_{(H_\rho^1(\Omega))', H_\rho^1(\Omega)} \\
&= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \frac{\mathcal{M}_\varepsilon(\psi)(\varepsilon(\xi - \mathbf{e}_1)) - \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)}{\varepsilon} \sum_{\mathbf{i} \in \mathbf{I}} \left[H^{(1)}(y - \mathbf{i}) - \frac{1}{2^{n-1}} \right] \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})).
\end{aligned}$$

Since $\sum_{\mathbf{i} \in \mathbf{I}} \left[H^{(1)}(y - \mathbf{i}) - \frac{1}{2^{n-1}} \right] = 0$ we obtain that

$$\left| \sum_{\mathbf{i} \in \mathbf{I}} \left[H^{(1)}(y - \mathbf{i}) - \frac{1}{2^{n-1}} \right] \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})) \right| \leq \sum_{\mathbf{i} \in \mathbf{I}} \left| \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi) \right|.$$

Taking into account the above equality and inequality we deduce that

$$\begin{aligned}
&< \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right), \psi >_{(H_\rho^1(\Omega))', H_\rho^1(\Omega)} \\
&= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \left| \frac{\mathcal{M}_\varepsilon(\psi)(\varepsilon(\xi - \mathbf{e}_1)) - \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)}{\varepsilon} \right| \left| \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi) \right| \\
&= \frac{1}{\varepsilon} \sum_{\mathbf{i} \in \mathbf{I}} \int_{\Omega} \left| \mathcal{M}_\varepsilon(\psi)(\cdot - \varepsilon\mathbf{e}_1) - \mathcal{M}_\varepsilon(\psi) \right| \left| \mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot + \varepsilon\mathbf{i}) - \mathcal{M}_\varepsilon(\phi_\varepsilon) \right| \\
&\leq \frac{C}{\varepsilon} \sum_{\mathbf{i} \in \mathbf{I}} \left\| \rho(\mathcal{M}_\varepsilon(\psi)(\cdot - \varepsilon\mathbf{e}_1) - \mathcal{M}_\varepsilon(\psi)) \right\|_{L^2(\Omega)} \left\| \frac{1}{\rho} (\mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot + \varepsilon\mathbf{i}) - \mathcal{M}_\varepsilon(\phi_\varepsilon)) \right\|_{L^2(\Omega)}.
\end{aligned}$$

Thanks to (2.18)₃ and (2.27)₃ we finally get

$$< \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right), \psi >_{(H_\rho^1(\Omega))', H_\rho^1(\Omega)} \leq C\varepsilon \|\phi_\varepsilon\|_{1/\rho} \|\psi\|_\rho$$

which leads to

$$\left\| \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right) \right\|_{L^\infty(Y; (H_\rho^1(\Omega))')} \leq C\varepsilon \|\phi_\varepsilon\|_{1/\rho}. \quad (3.10)$$

Besides we have

$$\int_\Omega \frac{\partial\phi}{\partial x_1}(x) \psi(x) dx = - \int_\Omega \phi(x) \frac{\partial\psi}{\partial x_1}(x) dx \leq C \|\phi/\rho\|_{L^2(\Omega)} \|\psi\|_\rho \leq C \|\phi\|_{1/\rho} \|\psi\|_\rho.$$

Hence $\left\| \varepsilon \frac{\partial\phi}{\partial x_1} \right\|_{(H_\rho^1(\Omega; \mathbb{R}^n))'} \leq C\varepsilon \|\phi\|_{1/\rho}$. This last estimate with (2.10)₂, (3.9) and (3.10) yield

$$\left\| \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right) - \frac{\partial\phi}{\partial x_1} \right\|_{L^\infty(Y; (H_\rho^1(\Omega))')} \leq C\varepsilon \|\phi_\varepsilon\|_{1/\rho}.$$

Proceeding in the same way we obtain the same estimates for the partial derivatives with respect to x_i , $i \in \{2, \dots, n\}$. Hence we get $\|\mathcal{T}_\varepsilon(\nabla\Phi) - \nabla\phi\|_{[L^\infty(Y; (H_\rho^1(\Omega))')^n]} \leq C\varepsilon \|\phi_\varepsilon\|_{1/\rho}$. Then, thanks to (3.8) the second estimate in (3.6) is proved. \square

4 Recalls about the classical periodic homogenization problem

We consider the homogenization problem

$$\phi^\varepsilon \in H_0^1(\Omega), \quad \int_\Omega A_\varepsilon(x) \nabla \phi^\varepsilon(x) \nabla \psi(x) dx = \int_\Omega f(x) \psi(x) dx, \quad \forall \psi \in H_0^1(\Omega), \quad (4.1)$$

where

- $A_\varepsilon(x) = A\left(\left\{\frac{x}{\varepsilon}\right\}\right)$ for a.e. $x \in \Omega$, here A is a square matrix belonging to $L^\infty(Y; \mathbb{R}^{n \times n})$, satisfying the condition of uniform ellipticity $c|\xi|^2 \leq A(y)\xi \cdot \xi \leq C|\xi|^2$ for a.e. $y \in Y$, with c and C strictly positive constants,
- $f \in L^2(\Omega)$.

We showed in [9] that

$$\mathcal{T}_\varepsilon(\nabla \phi^\varepsilon) \longrightarrow \nabla \Phi + \nabla_y \widehat{\phi} \quad \text{strongly in } L^2(\Omega \times Y; \mathbb{R}^n)$$

where $(\Phi, \widehat{\phi}) \in H_0^1(\Omega) \times L^2(\Omega; H_{per}^1(Y))$ is the solution of the problem of unfolding homogenization

$$\begin{aligned} & \forall (\Psi, \widehat{\psi}) \in H_0^1(\Omega) \times L^2(\Omega; H_{per}^1(Y)) \\ & \int_\Omega \int_Y A(y) \{ \nabla \Phi(x) + \nabla_y \widehat{\phi}(x, y) \} \{ \nabla \Psi(x) + \nabla_y \widehat{\psi}(x, y) \} dx dy = \int_\Omega f(x) \Psi(x) dx. \end{aligned}$$

The correctors χ_i , $i \in \{1, \dots, n\}$, are the solutions of the following variational problems:

$$\begin{aligned} \chi_i &\in H_{per}^1(Y), \quad \int_Y \chi_i = 0, \\ \int_Y A(y) \nabla_y (\chi_i(y) + y_i) \nabla_y \psi(y) dy &= 0, \quad \forall \psi \in H_{per}^1(Y). \end{aligned} \quad (4.2)$$

They allow to express $\widehat{\phi}$ in terms of the partial derivatives of Φ

$$\widehat{\phi} = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \chi_i \quad (4.3)$$

and to give the homogenized problem verified by Φ

$$\Phi \in H_0^1(\Omega), \quad \int_{\Omega} \mathcal{A} \nabla \Phi(x) \nabla \Psi(x) dx = \int_{\Omega} f(x) \Psi(x) dx, \quad \forall \Psi \in H_0^1(\Omega) \quad (4.4)$$

where (see [9])

$$\mathcal{A}_{ij} = \sum_{k,l=1}^n \int_Y a_{kl}(y) \frac{\partial(y_j + \chi_j(y))}{\partial y_l} \frac{\partial(y_i + \chi_i(y))}{\partial y_k} dy. \quad (4.5)$$

5 Error estimates with a Dirichlet condition

Theorem 5.1. *Let $(\phi^\varepsilon)_{\varepsilon>0}$ be a sequence of functions belonging to $H^1(\Omega)$ such that*

$$\operatorname{div}(A_\varepsilon \nabla \phi^\varepsilon) = 0 \quad \text{in } \Omega. \quad (5.1)$$

Setting $g_\varepsilon = \phi|_{\partial\Omega}^\varepsilon$ and $\phi_{g_\varepsilon} = \mathbf{T}(g_\varepsilon) \in H^1(\Omega)$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$ we have

$$\begin{aligned} \|\phi^\varepsilon\|_{H^1(\Omega)} &\leq C \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}, \quad \|\phi^\varepsilon - \phi_{g_\varepsilon}\|_{L^2(\Omega)} \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}, \\ \left\| \rho \left(\nabla \phi^\varepsilon - \nabla \phi_{g_\varepsilon} - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \nabla_y \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} &\leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \end{aligned} \quad (5.2)$$

Moreover we have

$$\|\phi^\varepsilon\|_\rho \leq C (\varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} + \|g_\varepsilon\|_{H^{-1/2}(\partial\Omega)}). \quad (5.3)$$

Proof. Step 1. We prove the first estimate in (5.2). From Section 7.1 we get

$$\|\phi_{g_\varepsilon}\|_{H^1(\Omega)} \leq C \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \quad \|\phi_{g_\varepsilon}\|_\rho \leq C \|g_\varepsilon\|_{H^{-1/2}(\partial\Omega)}. \quad (5.4)$$

We write (5.1) in the following weak form:

$$\begin{aligned} \phi^\varepsilon &= \overline{\phi}_\varepsilon + \phi_{g_\varepsilon}, \quad \overline{\phi}_\varepsilon \in H_0^1(\Omega) \\ \int_{\Omega} A_\varepsilon \nabla \overline{\phi}_\varepsilon \nabla v &= - \int_{\Omega} A_\varepsilon \nabla \phi_{g_\varepsilon} \nabla v \quad \forall v \in H_0^1(\Omega). \end{aligned} \quad (5.5)$$

The solution $\overline{\phi}_\varepsilon$ of the above variational problem satisfies

$$\|\overline{\phi}_\varepsilon\|_{H^1(\Omega)} \leq C \|\nabla \phi_{g_\varepsilon}\|_{L^2(\Omega; \mathbb{R}^n)}.$$

Hence, from (5.4)₁ and the above estimate we get the first inequality in (5.2).

Step 2. We prove the second estimate in (5.2). For every test function $v \in H_0^1(\Omega)$ we have

$$\int_{\Omega} A_\varepsilon \nabla \phi^\varepsilon \nabla v = 0. \quad (5.6)$$

Now, in order to obtain the L^2 error estimate we proceed as in the proof of the Theorem 3.2 in [15]. We first recall that for any $\phi \in H^1(\Omega)$ we have (see Lemma 2.3) for every $\varepsilon \leq \varepsilon_0 = \gamma_0/3\sqrt{n}$

$$\|\phi\|_{L^2(\tilde{\Omega}_{3c_0\sqrt{n\varepsilon}})} \leq C\varepsilon^{1/2} \|\phi\|_{H^1(\Omega)}.$$

Let U be a test function belonging to $H_0^1(\Omega) \cap H^2(\Omega)$. The above estimate yields

$$\|\nabla U\|_{L^2(\tilde{\Omega}_{3c_0\sqrt{n\varepsilon}}; \mathbb{R}^n)} \leq C\varepsilon^{1/2} \|U\|_{H^2(\Omega)} \quad (5.7)$$

which in turn with (2.12)-(2.13)-(2.14)₁ and (5.2)₁-(5.6) lead to

$$\left| \int_{\Omega \times Y} A(y) \mathcal{T}_\varepsilon(\nabla \phi^\varepsilon)(x, y) \nabla U(x) dx dy \right| \leq C\varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}. \quad (5.8)$$

The Theorem 2.3 in [15] gives an element $\widehat{\phi}_\varepsilon \in L^2(\Omega; H_{per}^1(Y))$ such that

$$\begin{aligned} \|\mathcal{T}(\nabla \phi^\varepsilon) - \nabla \phi^\varepsilon - \nabla_y \widehat{\phi}_\varepsilon\|_{[L^2(Y; (H^1(\Omega))')^n]^n} &\leq C\varepsilon^{1/2} \|\nabla \phi^\varepsilon\|_{L^2(\Omega; \mathbb{R}^n)} \\ &\leq C\varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \end{aligned} \quad (5.9)$$

The above inequality and (5.8) yield

$$\left| \int_{\Omega \times Y} A(\nabla \phi^\varepsilon + \nabla_y \widehat{\phi}_\varepsilon) \nabla U \right| \leq C\varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}. \quad (5.10)$$

We set

$$\forall x \in \mathbb{R}^n, \quad \rho_\varepsilon(x) = \inf \left\{ 1, \frac{\rho(x)}{\varepsilon} \right\}.$$

Now, we take $\overline{\chi} \in H_{per}^1(Y)$ and we consider the test function $u_\varepsilon \in H_0^1(\Omega)$ defined for a.e. $x \in \Omega$ by

$$u_\varepsilon(x) = \varepsilon \rho_\varepsilon(x) \mathcal{Q}_\varepsilon \left(\frac{\partial U}{\partial x_i} \right) (x) \overline{\chi} \left(\frac{x}{\varepsilon} \right).$$

Due to (2.24)₂ and (5.7) we get

$$\left\| \mathcal{Q}_\varepsilon \left(\frac{\partial U}{\partial x_i} \right) \nabla_y \overline{\chi} \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\tilde{\Omega}_{\sqrt{n\varepsilon}}; \mathbb{R}^n)} \leq C\varepsilon^{1/2} \|U\|_{H^2(\Omega)} \|\overline{\chi}\|_{H^1(Y)} \quad (5.11)$$

Then by a straightforward calculation and thanks to (2.24)₂-(2.25)₂ and (5.7)-(5.11) we obtain

$$\left\| \nabla u_\varepsilon - \mathcal{Q}_\varepsilon \left(\frac{\partial U}{\partial x_i} \right) \nabla_y \bar{\chi} \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \varepsilon^{1/2} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)}$$

which in turn with again (5.11) give

$$\|\nabla u_\varepsilon\|_{L^2(\tilde{\Omega}_{\sqrt{n\varepsilon}}; \mathbb{R}^n)} \leq C \varepsilon^{1/2} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)} \quad (5.12)$$

and then with (2.25)₁ that yields

$$\left\| \nabla u_\varepsilon - \mathcal{M}_\varepsilon \left(\frac{\partial U}{\partial x_i} \right) \nabla_y \bar{\chi} \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \varepsilon^{1/2} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)}.$$

In (5.6) we replace ∇u_ε with $\mathcal{M}_\varepsilon \left(\frac{\partial U}{\partial x_i} \right) \nabla_y \bar{\chi} \left(\frac{\cdot}{\varepsilon} \right)$; we continue using (2.12)-(2.13) and (5.2)₁-(5.12) to obtain

$$\left| \int_{\Omega \times Y} A(y) \mathcal{T}_\varepsilon(\nabla \phi^\varepsilon)(x, y) \mathcal{M}_\varepsilon \left(\frac{\partial U}{\partial x_i} \right) (x) \nabla_y \bar{\chi}(y) dx dy \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)}$$

which with (2.17)₂ and then (5.9) give

$$\left| \int_{\Omega \times Y} A(\nabla \phi^\varepsilon + \nabla_y \hat{\phi}_\varepsilon) \frac{\partial U}{\partial x_i} \nabla_y \bar{\chi} \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)}. \quad (5.13)$$

As in [15] we introduce the correctors $\bar{\chi}_i \in H_{per}^1(Y)$, $i \in \{1, \dots, n\}$, defined by

$$\int_Y A \nabla_y \psi \nabla_y (\bar{\chi}_i + y_i) = 0 \quad \forall \psi \in H_{per}^1(Y). \quad (5.14)$$

From (5.13) we get

$$\left| \int_{\Omega \times Y} A(\nabla \phi^\varepsilon + \nabla_y \hat{\phi}_\varepsilon) \nabla_y \left(\sum_{i=1}^n \frac{\partial U}{\partial x_i} \bar{\chi}_i \right) \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}$$

and from the definition (4.2) of the correctors χ_i we have

$$\int_{\Omega \times Y} A \left(\nabla \phi^\varepsilon + \sum_{i=1}^n \frac{\partial \phi^\varepsilon}{\partial x_i} \nabla_y \chi_i \right) \nabla_y \left(\sum_{j=1}^n \frac{\partial U}{\partial x_j} \bar{\chi}_j \right) = 0.$$

Thus

$$\left| \int_{\Omega \times Y} A \nabla_y \left(\hat{\phi}_\varepsilon - \sum_{i=1}^n \frac{\partial \phi^\varepsilon}{\partial x_i} \chi_i \right) \nabla_y \left(\sum_{j=1}^n \frac{\partial U}{\partial x_j} \bar{\chi}_j \right) \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}$$

and thanks to (5.14) we obtain

$$\left| \int_{\Omega \times Y} A \nabla_y \left(\widehat{\phi}_\varepsilon - \sum_{i=1}^n \frac{\partial \phi^\varepsilon}{\partial x_i} \chi_i \right) \nabla U \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}.$$

The above estimate, (5.10) and the expression (4.5) of the matrix \mathcal{A} yield

$$\left| \int_{\Omega} \mathcal{A} \nabla \phi^\varepsilon \nabla U \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}.$$

Finally, since we have $\int_{\Omega} \mathcal{A} \nabla \phi_{g_\varepsilon} \nabla v = 0$ for any $v \in H_0^1(\Omega)$, we deduce that

$$\forall U \in H_0^1(\Omega) \cap H^2(\Omega), \quad \left| \int_{\Omega} \mathcal{A} \nabla (\phi^\varepsilon - \phi_{g_\varepsilon}) \nabla U \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}.$$

Now, let $U_\varepsilon \in H_0^1(\Omega)$ be the solution of the following variational problem:

$$\int_{\Omega} \mathcal{A} \nabla v \nabla U_\varepsilon = \int_{\Omega} v (\phi^\varepsilon - \phi_{g_\varepsilon}), \quad \forall v \in H_0^1(\Omega).$$

Under the assumption on the boundary of Ω , we know that U_ε belongs to $H_0^1(\Omega) \cap H^2(\Omega)$ and satisfies $\|U_\varepsilon\|_{H^2(\Omega)} \leq C \|\phi^\varepsilon - \phi_{g_\varepsilon}\|_{L^2(\Omega)}$. Therefore, the second estimate in (5.2) is proved.

Step 3. We prove the third estimate in (5.2) and (5.3). The partial derivative $\frac{\partial \phi_{g_\varepsilon}}{\partial x_i}$ satisfies

$$\operatorname{div} \left(\mathcal{A} \nabla \left(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \in L^2(\Omega).$$

Thus, from Remark 7.7 and estimate (5.4)₂ we get

$$\left\| \rho \nabla \left(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \left\| \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right\|_{L^2(\Omega)} \leq C \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \quad (5.15)$$

Now, let U be in $H_0^1(\Omega)$, the function ρU belongs to $H_{1/\rho}^1(\Omega)$. Applying the Theorem 3.2 with the function ρU , there exists $\widehat{u}_\varepsilon \in L^2(\Omega; H_{per}^1(Y))$ such that

$$\|\mathcal{T}_\varepsilon(\nabla(\rho U)) - \nabla(\rho U) - \nabla_y \widehat{u}_\varepsilon\|_{L^2(Y; (H_{\rho}^1(\Omega; \mathbb{R}^n))')} \leq C \varepsilon \|\rho U\|_{H_{1/\rho}^1(\Omega)} \leq C \varepsilon \|U\|_{H^1(\Omega)}. \quad (5.16)$$

The above estimates (5.15) and (5.16) lead to

$$\left| \int_{\Omega \times Y} A \left(\nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \left(\mathcal{T}_\varepsilon(\nabla(\rho U)) - \nabla(\rho U) - \nabla_y \widehat{u}_\varepsilon \right) \right| \leq C \varepsilon \|U\|_{H^1(\Omega)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}$$

By definition of the correctors χ_i we have

$$\int_{\Omega \times Y} A \left(\nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \nabla_y \widehat{u}_\varepsilon = 0.$$

Besides, from the definitions of the function ϕ_{g_ε} and the homogenized matrix \mathcal{A} we have

$$0 = \int_{\Omega} \mathcal{A} \nabla \phi_{g_\varepsilon} \nabla (\rho U) = \int_{\Omega \times Y} A \left(\nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \nabla (\rho U).$$

The above inequality and equalities yield

$$\left| \int_{\Omega \times Y} A \left(\nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \mathcal{T}_\varepsilon(\nabla(\rho U)) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \quad (5.17)$$

We have

$$\nabla(\rho U) = \rho \left(\nabla U + \nabla \rho \frac{U}{\rho} \right)$$

and (3.2), therefore, since $U/\rho \in L^2(\Omega)$ and $\|U/\rho\|_{L^2(\Omega)} \leq C \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)}$ we get

$$\left\| \mathcal{T}_\varepsilon(\nabla(\rho U)) - \rho \mathcal{T}_\varepsilon \left(\nabla U + \nabla \rho \frac{U}{\rho} \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\varepsilon \left\| \nabla U + \nabla \rho \frac{U}{\rho} \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\varepsilon \|U\|_{H^1(\Omega)}.$$

From (5.17) and the above inequalities we deduce that

$$\left| \int_{\Omega \times Y} A \left(\rho \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \rho \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \mathcal{T}_\varepsilon \left(\nabla U + \nabla \rho \frac{U}{\rho} \right) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}.$$

We recall that $\rho \nabla \phi_{g_\varepsilon} \in H_0^1(\Omega; \mathbb{R}^n)$, hence from (2.14)₂, (2.17)₁ and (5.15) we get

$$\begin{aligned} & \left| \int_{\Omega \times Y} A \left(\rho \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \rho \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \mathcal{T}_\varepsilon \left(\nabla U + \nabla \rho \frac{U}{\rho} \right) \right. \\ & \left. - \int_{\Omega \times Y} A \left(\mathcal{T}_\varepsilon(\rho \nabla \phi_{g_\varepsilon}) + \sum_{i=1}^n \mathcal{M}_\varepsilon \left(\rho \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \nabla_y \chi_i \right) \mathcal{T}_\varepsilon \left(\nabla U + \nabla \rho \frac{U}{\rho} \right) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

Then, transforming by inverse unfolding we obtain

$$\left| \int_{\hat{\Omega}_\varepsilon} A_\varepsilon \left(\rho \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \mathcal{M}_\varepsilon \left(\rho \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \nabla_y \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right) \left(\nabla U + \nabla \rho \frac{U}{\rho} \right) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}.$$

Now, thanks to (2.28) and (5.15) we get

$$\left| \int_{\Omega} A_\varepsilon \rho \left(\nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \mathcal{M}_\varepsilon \left(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \nabla_y \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right) \left(\nabla U + \nabla \rho \frac{U}{\rho} \right) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}.$$

Then using (2.29)₁ it leads to

$$\left| \int_{\Omega} A_\varepsilon \left(\nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \nabla_y \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right) \nabla(\rho U) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}.$$

We recall that $\int_{\Omega} A_{\varepsilon} \nabla \phi^{\varepsilon} \nabla (\rho U) = 0$. We choose $U = \rho \left(\phi^{\varepsilon} - \phi_{g_{\varepsilon}} - \varepsilon \sum_{i=1}^n \mathcal{Q}_{\varepsilon} \left(\frac{\partial \phi_{g_{\varepsilon}}}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right)$ which belongs to $H_0^1(\Omega)$. Due to the second estimate in (5.2), the third one in (5.2) follows immediately. The estimates (5.3) are the consequences of (5.2)₂, (5.2)₃ and (5.4)₂. \square

Corollary 5.2. *Let $(\phi^{\varepsilon})_{\varepsilon > 0}$ be a sequence of functions belonging to $H^1(\Omega)$ and satisfying (5.1). We set $g_{\varepsilon} = \phi_{\partial\Omega}^{\varepsilon}$, if we have*

$$g_{\varepsilon} \rightharpoonup g \quad \text{weakly in} \quad H^{1/2}(\partial\Omega)$$

then we obtain

$$\begin{aligned} \phi^{\varepsilon} &\rightharpoonup \phi_g \quad \text{weakly in} \quad H^1(\Omega), \\ \phi^{\varepsilon} - \phi_g - \varepsilon \sum_{i=1}^n \mathcal{Q}_{\varepsilon} \left(\frac{\partial \phi_g}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) &\longrightarrow 0 \quad \text{strongly in} \quad H_{\rho}^1(\Omega). \end{aligned} \quad (5.18)$$

Moreover, if

$$g_{\varepsilon} \longrightarrow g \quad \text{strongly in} \quad H^{1/2}(\partial\Omega) \quad (5.19)$$

then we have

$$\phi^{\varepsilon} - \phi_g - \varepsilon \sum_{i=1}^n \mathcal{Q}_{\varepsilon} \left(\frac{\partial \phi_g}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \longrightarrow 0 \quad \text{strongly in} \quad H^1(\Omega). \quad (5.20)$$

Proof. Thanks to (5.2)₁ the sequence $(\phi^{\varepsilon})_{\varepsilon > 0}$ is uniformly bounded in $H^1(\Omega)$. Then, due to Lemma 7.1 and Remark 7.7 we get

$$\|\phi_g - \phi_{g_{\varepsilon}}\|_{\rho} \leq C \|g - g_{\varepsilon}\|_{H^{-1/2}(\partial\Omega)}$$

which with (5.2)₂ (resp. (5.2)₃) give the convergence (5.18)₁ (resp. (5.18)₂).

Under the assumption (5.19), we use (7.1) and we proceed as in the proof of Theorem 6.1 of [9] to obtain the strong convergence (5.20). \square

Theorem 5.3. *Let ϕ^{ε} be the solution of the following homogenization problem:*

$$-\operatorname{div}(A_{\varepsilon} \nabla \phi^{\varepsilon}) = f \quad \text{in} \quad \Omega, \quad \phi^{\varepsilon} = g \quad \text{on} \quad \partial\Omega$$

where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. We have

$$\begin{aligned} \|\phi^{\varepsilon} - \Phi\|_{L^2(\Omega)} &\leq C \{ \varepsilon \|f\|_{L^2(\Omega)} + \varepsilon^{1/2} \|g\|_{H^{1/2}(\partial\Omega)} \}, \\ \left\| \rho \left(\nabla \phi^{\varepsilon} - \nabla \Phi - \sum_{i=1}^n \mathcal{Q}_{\varepsilon} \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} &\leq C \{ \varepsilon \|f\|_{L^2(\Omega)} + \varepsilon^{1/2} \|g\|_{H^{1/2}(\partial\Omega)} \} \end{aligned}$$

where Φ is the solution of the homogenized problem

$$-\operatorname{div}(\mathcal{A} \nabla \Phi) = f \quad \text{in} \quad \Omega, \quad \Phi = g \quad \text{on} \quad \partial\Omega.$$

Moreover we have

$$\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \longrightarrow 0 \quad \text{strongly in } H^1(\Omega). \quad (5.21)$$

Proof. Let $\tilde{\phi}^\varepsilon$ be the solution of the homogenization problem

$$\tilde{\phi}^\varepsilon \in H_0^1(\Omega), \quad -\operatorname{div}(A_\varepsilon \nabla \tilde{\phi}^\varepsilon) = f \quad \text{in } \Omega$$

and $\tilde{\Phi}$ the solution of the homogenized problem

$$\tilde{\Phi} \in H_0^1(\Omega), \quad -\operatorname{div}(\mathcal{A} \nabla \tilde{\Phi}) = f \quad \text{in } \Omega.$$

The Theorem 3.2 in [15] gives the following estimate:

$$\|\tilde{\phi}^\varepsilon - \tilde{\Phi}\|_{L^2(\Omega)} + \left\| \rho \nabla \left(\tilde{\phi}^\varepsilon - \tilde{\Phi} - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \tilde{\Phi}}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\varepsilon \|f\|_{L^2(\Omega)} \quad (5.22)$$

while the Theorem 4.1 in [14] gives

$$\left\| \tilde{\phi}^\varepsilon - \tilde{\Phi} - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \tilde{\Phi}}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right\|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \|f\|_{L^2(\Omega)}. \quad (5.23)$$

Then $\phi^\varepsilon - \tilde{\phi}^\varepsilon$ satisfies

$$\operatorname{div}(A_\varepsilon \nabla (\phi^\varepsilon - \tilde{\phi}^\varepsilon)) = 0 \quad \text{in } \Omega, \quad \phi^\varepsilon - \tilde{\phi}^\varepsilon = g \quad \text{on } \partial\Omega.$$

Thanks to the inequalities (5.2) and (5.22) we deduce the estimates of the theorem. The strong convergence (5.21) is a consequence of (5.23) and the strong convergence (5.20) after having observed that $\Phi - \tilde{\Phi} = \phi_g$. \square

Remark 5.4. In Theorem 5.3, if $g \in H^{3/2}(\partial\Omega)$ then in the estimates therein, we can replace $\varepsilon^{1/2} \|g\|_{H^{1/2}(\partial\Omega)}$ with $\varepsilon \|g\|_{H^{3/2}(\partial\Omega)}$. Moreover we have the following H^1 -global error estimate:

$$\left\| \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right\|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \{ \|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)} \}.$$

6 A first result with strongly oscillating boundary data

Now, we consider the solution ϕ^ε of the following homogenization problem:

$$\begin{aligned} \operatorname{div}(A_\varepsilon \nabla \phi^\varepsilon) &= 0 \quad \text{in } \Omega \\ \phi^\varepsilon &= g_\varepsilon \quad \text{on } \partial\Omega \end{aligned} \quad (6.1)$$

where $g_\varepsilon \in H^{1/2}(\partial\Omega)$. As a consequence of the Theorem 5.1 we first obtain the following theorem:

Theorem 6.1. *Let ϕ^ε be the solution of the Problem (6.1). If we have*

$$g_\varepsilon \rightharpoonup g \quad \text{weakly in} \quad H^{-1/2}(\partial\Omega)$$

and

$$\varepsilon^{1/2} g_\varepsilon \longrightarrow 0 \quad \text{strongly in} \quad H^{1/2}(\partial\Omega) \quad (6.2)$$

then

$$\phi^\varepsilon \rightharpoonup \phi_g \quad \text{weakly in} \quad H_\rho^1(\Omega). \quad (6.3)$$

Furthermore, if we have

$$g_\varepsilon \longrightarrow g \quad \text{strongly in} \quad H^{-1/2}(\partial\Omega)$$

then

$$\phi^\varepsilon - \phi_g - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \longrightarrow 0 \quad \text{strongly in} \quad H_\rho^1(\Omega). \quad (6.4)$$

Proof. Due to (5.3) the sequence $(\phi^\varepsilon)_{\varepsilon>0}$ is strictly bounded in $H_\rho^1(\Omega)$. From the estimates (5.2)₃ and (5.4)₂ we get

$$\left\| \phi^\varepsilon - \phi_{g_\varepsilon} - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \right\|_{H_\rho^1(\Omega)} \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}.$$

Then, using the variational problem (7.4) and (5.4)₂ we obtain

$$\phi_{g_\varepsilon} \rightharpoonup \phi_g \quad \text{weakly in} \quad H_\rho^1(\Omega).$$

Besides we have $\varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \chi_i \left(\frac{\cdot}{\varepsilon} \right) \rightharpoonup 0$ weakly in $H_\rho^1(\Omega)$. The weak convergence (6.3) is proved.

In the case $g_\varepsilon \longrightarrow g$ strongly in $H^{-1/2}(\partial\Omega)$, (7.3) and (7.7) lead to

$$\|\phi_{g_\varepsilon} - \phi_g\|_{H_\rho^1(\Omega)} \leq C \|g_\varepsilon - g\|_{H^{-1/2}(\partial\Omega)}.$$

Hence with (2.29)₂ they yield the strong convergence (6.4). \square

In a forthcoming paper we will show that in both cases (weak or strong convergence of g_ε towards g in $H^{-1/2}(\partial\Omega)$) the assumption (6.2) is essential in order to obtain at least (6.3).

7 Appendix

7.1 An operator from $H^{-1/2}(\partial\Omega)$ into $L^2(\Omega)$

In this section Ω is an open bounded set with a $\mathcal{C}^{1,1}$ boundary or an open bounded convex polygon ($n = 2$) or polyhedral ($n = 3$). Let \mathcal{A} be a $n \times n$ definite positive constant matrix (e.g. the matrix \mathcal{A} given by (4.5)). For every couple $(\psi, \Psi) \in [\mathcal{C}^\infty(\overline{\Omega})]^2$, integrating by parts over Ω gives

$$\int_{\Omega} \mathcal{A} \nabla \psi(x) \nabla \Psi(x) dx = - \int_{\Omega} \psi(x) \operatorname{div}(\mathcal{A}^T \nabla \Psi)(x) dx + \int_{\partial\Omega} \psi(x) (\mathcal{A}^T \nabla \Psi)(x) \cdot \nu(x) d\sigma.$$

The space $\mathcal{C}^\infty(\overline{\Omega})$ being dense in $H^1(\Omega)$ and $H^2(\Omega)$, hence the above equality holds true for any $\psi \in H^1(\Omega)$ and any $\Psi \in H^2(\Omega)$. Now, let g be in $H^{1/2}(\partial\Omega)$, there exists one $\phi_g \in H^1(\Omega)$ such that

$$\operatorname{div}(\mathcal{A} \nabla \phi_g) = 0 \quad \text{in } \Omega, \quad \phi_g = g \quad \text{on } \partial\Omega$$

and we have

$$\|\phi_g\|_{H^1(\Omega)} \leq C \|g\|_{H^{1/2}(\partial\Omega)}. \quad (7.1)$$

Besides, for any function $\Psi \in H_0^1(\Omega) \cap H^2(\Omega)$ we get

$$\int_{\Omega} \phi_g(x) \operatorname{div}(\mathcal{A}^T \nabla \Psi)(x) dx = \int_{\partial\Omega} g(x) (\mathcal{A}^T \nabla \Psi)(x) \cdot \nu(x) d\sigma. \quad (7.2)$$

Under the assumption on Ω the function $\Psi(g)$ defined by

$$\Psi(g) \in H_0^1(\Omega), \quad \operatorname{div}(\mathcal{A}^T \nabla \Psi(g)) = \phi_g \quad \text{in } \Omega$$

also belongs to $H^2(\Omega)$ and satisfies

$$\|\Psi(g)\|_{H^2(\Omega)} \leq C \|\phi_g\|_{L^2(\Omega)}.$$

Taking $\Psi = \Psi(g)$ in the above equality (7.2) we obtain

$$\begin{aligned} \int_{\Omega} |\phi_g(x)|^2 dx &= \int_{\partial\Omega} g(x) (\mathcal{A}^T \nabla \Psi(g)(x)) \cdot \nu(x) d\sigma \leq \|g\|_{H^{-1/2}(\partial\Omega)} \|(\mathcal{A}^T \nabla \Psi(g)) \cdot \nu\|_{H^{1/2}(\partial\Omega)} \\ &\leq C \|g\|_{H^{-1/2}(\partial\Omega)} \|\Psi(g)\|_{H^2(\Omega)}. \end{aligned}$$

This leads to

$$\|\phi_g\|_{L^2(\Omega)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)}. \quad (7.3)$$

We denote by \mathbf{T} the operator from $H^{1/2}(\partial\Omega)$ into $H^1(\Omega)$ which associates to $g \in H^{1/2}(\partial\Omega)$ the function $\phi_g \in H^1(\Omega)$. Due to (7.3), this operator admits an extension (still denoted \mathbf{T}) from $H^{-1/2}(\partial\Omega)$ into $L^2(\Omega)$ and we have

$$\forall g \in H^{-1/2}(\partial\Omega), \quad \|\mathbf{T}(g)\|_{L^2(\Omega)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)}.$$

Hence, for $g \in H^{-1/2}(\partial\Omega)$, we also denote $\phi_g = \mathbf{T}(g)$. This function is the "very weak" solution of the problem

$$\phi_g \in L^2(\Omega), \quad \operatorname{div}(\mathcal{A}\nabla\phi_g) = 0 \quad \text{in } \Omega, \quad \phi_g = g \quad \text{on } \partial\Omega$$

or the solution of the following:

$$\begin{aligned} \phi_g &\in L^2(\Omega), \\ \int_{\Omega} \phi_g(x) \operatorname{div}(\mathcal{A}^T \nabla \psi(x)) dx &= \langle g, (\mathcal{A}^T \nabla \psi) \cdot \nu \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \\ \forall \psi &\in H_0^1(\Omega) \cap H^2(\Omega). \end{aligned} \tag{7.4}$$

Lemma 7.1. *The operator \mathbf{T} is a bicontinuous linear operator from $H^{-1/2}(\partial\Omega)$ onto*

$$\mathbf{H} = \left\{ \phi \in L^2(\Omega) \mid \operatorname{div}(\mathcal{A}\nabla\phi) = 0 \quad \text{in } \Omega \right\}.$$

There exists a constant $C \geq 1$ such that

$$\forall g \in H^{-1/2}(\partial\Omega), \quad \frac{1}{C} \|g\|_{H^{-1/2}(\partial\Omega)} \leq \|\mathbf{T}(g)\|_{L^2(\Omega)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)}. \tag{7.5}$$

Proof. Let ϕ be in \mathbf{H} we are going to prove that there exists an element $g \in H^{-1/2}(\partial\Omega)$ such that $\mathbf{T}(g) = \phi$. To do that, we consider a continuous linear lifting operator \mathbf{R} from $H^{1/2}(\partial\Omega)$ into $H_0^1(\Omega) \cap H^2(\Omega)$ satisfying for any $h \in H^{1/2}(\partial\Omega)$

$$\begin{aligned} \mathbf{R}(h) &\in H_0^1(\Omega) \cap H^2(\Omega), \\ \mathcal{A}^T \nabla \mathbf{R}(h)|_{\partial\Omega} \cdot \nu &= h \quad \text{on } \partial\Omega, \\ \|\mathbf{R}(h)\|_{H^2(\Omega)} &\leq C \|h\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

The map $h \mapsto \int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \mathbf{R}(h))$ is a continuous linear form defined over $H^{1/2}(\partial\Omega)$. Thus, there exists $g \in H^{-1/2}(\partial\Omega)$ such that

$$\int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \mathbf{R}(h)) = \langle g, h \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}. \tag{7.6}$$

Since $\phi \in \mathbf{H}$, we deduce that for any $\psi \in \mathcal{C}_0^\infty(\Omega)$ we have $\int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \psi) = 0$. Therefore, for any $\psi \in H_0^2(\Omega)$ we have $\int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \psi) = 0$. Taking into account (7.6) we get

$$\int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \psi) = \langle g, (\mathcal{A}^T \nabla \psi) \cdot \nu \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \quad \forall \psi \in H_0^1(\Omega) \cap H^2(\Omega).$$

It yields $\phi = \phi_g$ and then (7.5). \square

Remark 7.2. *It is well known (see e.g. [17]) that every function $\phi \in \mathbf{H}$ also belongs to $H_\rho^1(\Omega)$ and verifies*

$$\|\phi\|_\rho \leq C \|\phi\|_{L^2(\Omega)}. \tag{7.7}$$

References

- [1] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23 (1992), pp. 1482-1518.
- [2] G. Allaire and M. Amar. Boundary layer tails in periodic homogenization, *ESAIM: Control, Optimization and Calc. of Variations*, Vol. 4, (1999), 209-243.
- [3] M. Avellaneda and F.-H. Lin. Homogenization of elliptic problems with L^p boundary data, *Appl. Math. Optim.*, 15 (1987), 93-107.
- [4] A. Bensoussan, J.-L. Lions and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. North Holland, Amsterdam, 1978.
- [5] D. Blanchard, A. Gaudiello and G. Griso. Junction of a periodic family of elastic rods with a thin plate. II, *J. Math. Pures Appl.* vol. 88 (2007), 149-190.
- [6] A. Blasselle and G. Griso. Mechanical modeling of the skin, *Asymptot. Anal.*, 74 (2011), 3-4, 167-198.
- [7] A. Bossavit, G. Griso, and B. Miara. Modeling of periodic electro-magnetic structures. *Bianisotropic materials with memory effects*, *J.M.P.A.*, vol. 84 (2005), 819-850.
- [8] H. Brezis. *Analyse fonctionnelle. Théorie et applications*, Masson, Paris, 1983.
- [9] D. Cioranescu, A. Damlamian and G. Griso. The periodic unfolding method in homogenization, *SIAM J. of Math. Anal.* Vol. 40, 4 (2008), 1585-1620.
- [10] D. Cioranescu, A. Damlamian, G. Griso and D. Onofrei. The periodic unfolding method for perforated domains and Neumann sieve models, *J. Math. Pures Appl.*, Vol. 89 (2008), 248-277.
- [11] D. Cioranescu, A. Damlamian, P. Donato, G. Griso and R. Zaki. The periodic unfolding method in domains with holes, *SIAM J. of Math. Anal.* Vol. 44, 2 (2012), 718-760.
- [12] D. Cioranescu and P. Donato. *An Introduction to Homogenization*. Oxford Lecture Series in Mathematics and its Applications 17, Oxford University Press, 1999.
- [13] G. Griso. Estimation d'erreur et éclatement en homogénéisation périodique. *C. R. Acad. Sci. Paris, Ser. I* 335 (2002), 333-336.
- [14] G. Griso. Error estimate and unfolding for periodic homogenization, *Asymptot. Anal.*, 40 (2004), 269-286.
- [15] G. Griso. Interior error estimates for periodic homogenization, *Analysis and Applications*, Vol. 4 (2006), 61-79.

- [16] G. Griso. Asymptotic behavior of structures made of plates, *Analysis Appl.* 3 (4) (2005) 325-356.
- [17] G. Griso. Decomposition of displacements of thin structures. *J. Math. Pures Appl.* 89 (2008), 199-233.
- [18] S. Kesavan. Homogenization of elliptic eigenvalue problems: Part 1. *Appl. Math. Optim.*, Vol. 5, 1 (1979), 153-167.
- [19] J.L. Lions and E. Magenes. *Problèmes aux limites non homogènes*, Dunod, Paris, 1968.
- [20] S. Moskow and M. Vogelius. First-order corrections to the homogenised eigenvalues of a periodic composite medium. A convergence proof, *Proceedings of the Royal Society of Edimburgh*, 127A (1997), 1263-1299.
- [21] D. Onofrei, B. Vernescu. Error estimates for periodic homogenization with non-smooth coefficients, *Asymptot. Anal.*, Vol. 54 (2007), 103-123.